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A short note on the asymptotic stability of an oscillation-free eikonal splitting method

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ABSTRACT

Different beam propagation methods (BPMs) have been fundamental in modern electromagnetical wave simulations. Challenges of the numerical strategy include the computational efficiency and stability, in particular when highly oscillatory optical waves are present. This paper concerns an eikonal splitting BPM scheme for two-dimensional paraxial Helmholtz equations together with transparent boundary conditions in slowly varying envelope approximations of active laser beams. It is shown that the finite difference method investigated is not only oscillation-free, but also asymptotically stable. This ensures the high efficiency and applicability in highly oscillatory wave applications.

1. Introduction

Let $\mathcal{D} = \{(x, y, z) : a \le x \le b, c \le y \le d, z \ge z_0\}$ be an active beam domain considered, where z is the beam propagation direction, x, y are dimensional directions perpendicular to the light, a, b, c, d are real constants and z_0 is the initial beam location. In a slowly varying envelope approximation of the light beam propagation, the light movement can be ideally described by the two-dimensional paraxial Helmholtz equation

$$2i\omega\frac{\partial E}{\partial z} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}, \quad (x, y, z) \in \mathcal{D}, \tag{1.1}$$

where $i = \sqrt{-1}$, $\omega = 2\pi/\lambda$ is the wave number, λ is the wavelength and E is the complex envelope of the wave function anticipated [1–4].

Let E_0 be the initial envelope function, that is,

$$E(x, y, z_0) = E_0(x, y), \quad a \le x \le b, \ c \le y \le d. \tag{1.2}$$

Typical continuous transparent boundary conditions [5,3] for (1.1) include

$$\frac{\partial E}{\partial x}(a, y, z) = K_a e^{-i(1+V_a z)} \frac{d}{dz}\int_{z_0}^{z} \frac{E(a, y, s)e^{iV_a s}}{\sqrt{z-s}} ds, \tag{1.3}$$

$$\frac{\partial E}{\partial x}(b, y, z) = -K_b e^{-i(1+V_b z)} \frac{d}{dz}\int_{z_0}^{z} \frac{E(b, y, s)e^{iV_b s}}{\sqrt{z-s}} ds, \tag{1.4}$$

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$$\frac{\partial E}{\partial y}(x, c, z) = K_c e^{-i(1+V_c z)} \frac{d}{dz} \int_{z_0}^{z} \frac{E(x, c, s)e^{iV_c s}}{\sqrt{z-s}} ds, \tag{1.5}$$

$$\frac{\partial E}{\partial y}(x, d, z) = -K_d e^{-i(1+V_d z)} \frac{d}{dz} \int_{z_0}^{z} \frac{E(x, d, s)e^{iV_d s}}{\sqrt{z-s}} ds, \tag{1.6}$$

where K_a, K_b, K_c, K_d and V_a, V_b, V_c, V_d are given constant parameters. Observe that

$$\begin{aligned}\int_{z_0}^{z} \frac{E(a, y, s)e^{iV_a s}}{\sqrt{z-s}} ds &= E(a, y, \xi_a)e^{iV_a \xi_a} \int_{z_0}^{z} \frac{1}{\sqrt{z-s}} ds \\ &= 2E(a, y, \xi_a)e^{iV_a \xi_a}\sqrt{z-z_0}, \quad z_0 < \xi_a < z.\end{aligned}$$

Therefore the nonlocal condition (1.3) may be approximated by

$$\frac{\partial E}{\partial x}(a, y, z) \approx K_a e^{-i(1+V_a(z-\xi_a))} \frac{E(a, y, \xi_a)}{\sqrt{z-z_0}}, \quad z_0 < \xi_a < z, \tag{1.7}$$

if ξ_a is frozen. By the same token, we may replace (1.4)–(1.6) by

$$\frac{\partial E}{\partial x}(b, y, z) \approx -K_b e^{-i(1+V_b(z-\xi_b))} \frac{E(b, y, \xi_b)}{\sqrt{z-z_0}}, \quad z_0 < \xi_b < z, \tag{1.8}$$

$$\frac{\partial E}{\partial y}(x, c, z) \approx K_c e^{-i(1+V_c(z-\xi_c))} \frac{E(x, c, \xi_c)}{\sqrt{z-z_0}}, \quad z_0 < \xi_c < z, \tag{1.9}$$

$$\frac{\partial E}{\partial y}(x, d, z) \approx -K_d e^{-i(1+V_d(z-\xi_d))} \frac{E(x, d, \xi_d)}{\sqrt{z-z_0}}, \quad z_0 < \xi_d < z. \tag{1.10}$$

2. Eikonal splitting approximation

Since $10^5 \leq \omega \leq 10^8$ in optical applications [1,3], E is highly oscillatory. This poses a challenge to the design of a practically efficient and reliable BPM for solving (1.1), (1.2), (1.7)–(1.10), since mesh step sizes cannot be unrealistically small [6]. For this, based on the eikonal transformation [1,7],

$$E(x, y, z) = u(x, y, z)e^{i\omega v(x,y,z)}, \quad 0 < \sigma_0 \leq u \leq \sigma_1. \tag{2.1}$$

Guha et al. [8,4] introduced the following nonlinear ray system replacing (1.1),

$$\frac{\partial w}{\partial z} = P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + g, \quad (x, y, z) \in \mathcal{D}, \tag{2.2}$$

where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

and

$$\alpha = \frac{u}{2}, \qquad \beta = -\frac{1}{2\omega^2 u}, \qquad g_1 = u_x v_x + u_y v_y, \qquad g_2 = \frac{1}{2}\left[(v_x)^2 + (v_y)^2\right]. \tag{2.3}$$

Note that the amplitude and phase functions u, v are real and non-oscillatory. The condition (1.2) can be reformulated to yield

$$u(x, y, z_0) = u_0(x, y), \qquad v(x, y, z_0) = v_0(x, y), \quad a \leq x \leq b,\ c \leq y \leq d. \tag{2.4}$$

For a sufficiently large integer N, we introduce uniform step sizes $h_x = (b-a)/(N+1)$, $h_y = (d-c)/(N+1)$ and $h_z \ll 1$. Define the mesh region $\mathcal{D}_h = \{(x_m, y_n, z_k) : x_m = a + mh_x, y_n = c + nh_y, z_k = z_0 + kh_z, m, n = 0, 1, \ldots, N+1; k \geq 0\}$ superimposing $\mathcal{D}$. We denote a mesh function $\phi_{m,n}^{(k)} = \phi(x_m, y_n, z_k)$. Due to the strong nonlinearity of (2.2) and (2.3), instead of a conventional implicit scheme, we adopt the following splitting method [6] for (2.2)–(2.4):

$$w_{m,n}^{(k+1/2)} - w_{m,n}^{(k)} = P_{m,n}^{(k)}\left(\mu\delta_x^2 w_{m,n}^{(k+1/2)} + \eta\delta_y^2 w_{m,n}^{(k)}\right) + \frac{h_z}{2} g_{m,n}^{(k)}, \tag{2.5}$$

$$w_{m,n}^{(k+1)} - w_{m,n}^{(k+1/2)} = P_{m,n}^{(k+1/2)}\left(\mu\delta_x^2 w_{m,n}^{(k+1/2)} + \eta\delta_y^2 w_{m,n}^{(k+1)}\right) + \frac{h_z}{2} g_{m,n}^{(k+1/2)}, \tag{2.6}$$

$$w_{m,n}^{(0)} = \left(u_{m,n}^{(0)}, v_{m,n}^{(0)}\right)^T, \quad m, n = 1, 2, \ldots, N;\ k = 0, 1, \ldots, \tag{2.7}$$

where δ_x^2, δ_y^2 are standard central partial difference operators and $\mu = h_z/(2h_x^2)$, $\eta = h_z/(2h_y^2)$ are dimensional Courant numbers.

Further, in a BPM configuration [1], we may replace (1.7)–(1.10) by following ready-to-use nonhomogeneous Robin conditions:

$$\frac{\partial w}{\partial x}(a, y, z) - q_a(y, z)w(a, y, z) = -p_a(y, z),$$
$$\frac{\partial w}{\partial x}(b, y, z) + q_b(y, z)w(b, y, z) = p_b(y, z),$$
$$\frac{\partial w}{\partial y}(x, c, z) - q_c(x, z)w(x, c, z) = -p_c(x, z),$$
$$\frac{\partial w}{\partial y}(x, d, z) + q_d(x, z)w(x, d, z) = p_d(x, z),$$

where $p_{(\cdot)}, q_{(\cdot)}$ are continuous on their respective domains. To match the second order accuracy of (2.5)–(2.7), we again utilize central difference for approximating partial derivatives in above conditions. This leads to

$$w(a - h_x, y, z) = w(a + h_x, y, z) - 2h_x q_a w(a, y, z) + 2h_x p_a, \tag{2.8}$$
$$w(b + h_x, y, z) = w(b - h_x, y, z) - 2h_x q_b w(b, y, z) + 2h_x p_b, \tag{2.9}$$
$$w(x, c - h_y, z) = w(x, c + h_y, z) - 2h_y q_c w(x, c, z) + 2h_y p_c, \tag{2.10}$$
$$w(x, d + h_y, z) = w(x, d - h_y, z) - 2h_y q_d w(x, d, z) + 2h_y p_d. \tag{2.11}$$

It follows subsequently that (2.5)–(2.7), (2.8)–(2.11) can be comprised to yield the following eikonal splitting method [8,2,4],

$$\left(I - \mu P^{(k)}\right) w^{(k+1/2)} = \left(I + \eta Q^{(k)}\right) w^{(k)} + f_1^{(k)}, \tag{2.12}$$
$$\left(I - \eta Q^{(k+1/2)}\right) w^{(k+1)} = \left(I + \mu P^{(k+1/2)}\right) w^{(k+1/2)} + f_2^{(k+1/2)}, \tag{2.13}$$

where $I \in \mathbb{R}^{2N^2 \times 2N^2}$ is an identity matrix and for $r \in \{k, k + 1/2\}$,

$$P^{(r)} = R^{(r)} \left(I_N \otimes S_{q_a,q_b,h_x} \otimes I_2\right), \qquad Q^{(r)} = R^{(r)} \left(S_{q_c,q_d,h_y} \otimes I_N \otimes I_2\right),$$

in which

$$R^{(r)} = \operatorname{diag}\left(P_1^{(r)}, P_2^{(r)}, \dots, P_N^{(r)}\right), \tag{2.14}$$
$$P_j^{(r)} = \operatorname{diag}\left(P_{1,j}^{(r)}, P_{2,j}^{(r)}, \dots, P_{N,j}^{(r)}\right), \quad j = 1, 2, \dots, N,$$

$I_N \in \mathbb{R}^{N\times N}$, $I_2 \in \mathbb{R}^{2\times 2}$ are identity matrices and $S_{\xi,\zeta,\gamma} \in \mathbb{R}^{N\times N}$ is a tridiagonal matrix of the form

$$S(\xi, \zeta, \gamma) = \begin{pmatrix} -2(\xi\gamma + 1) & 2 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2(\zeta\gamma + 1) \end{pmatrix}.$$

Vectors $f_1^{(k)}, f_2^{(k+1/2)}$ are independent of solutions $w^{(k+1/2)}, w^{(k+1)}$, respectively. Clearly, the decomposed BPM (2.12), (2.13) is of second order and implicit.

3. Linear asymptotic stability

Definition 3.1. Let A be the amplifying matrix of a BPM scheme for solving the oscillatory problem (1.1)–(1.6) together with a high wave number ω. We say that the method is order σ asymptotically stable if there exists a constant $\sigma > 0$ such that

$$\|A\|_2 = 1 + O\left(1/\omega^{\sigma}\right) \quad \text{as } \omega \to \infty. \tag{3.1}$$

If (3.1) holds for all step sizes h_x, h_y and h_z then the asymptotic stability is unconditional.

Lemma 3.2. *We have*

$$\|I_N \otimes S_{q_a,q_b,h_x} \otimes I_2\|_2 \le \|S_{q_a,q_b,h_x}\|_2 \le c_1,$$
$$\|S_{q_c,q_d,h_y} \otimes I_N \otimes I_2\|_2 \le \|S_{q_c,q_d,h_y}\|_2 \le c_2$$

for some positive constants c_1, c_2.

Proof. We only need to show the first inequality since the other one is similar. Based on properties of the Kronecker product,

$$\begin{aligned}(I_N \otimes S_{q_a,q_b,h_x} \otimes I_2)(I_N \otimes S_{q_a,q_b,h_x} \otimes I_2)^T &= (I_N \otimes S_{q_a,q_b,h_x} \otimes I_2)\left(I_N^T \otimes S^T_{q_a,q_b,h_x} \otimes I_2^T\right)\\ &= I_N \otimes (S_{q_a,q_b,h_x} \otimes I_2)\left(S^T_{q_a,q_b,h_x} \otimes I_2\right)\\ &= I_N \otimes S_{q_a,q_b,h_x} S^T_{q_a,q_b,h_x} \otimes I_2.\end{aligned}$$

Since

$$\mathrm{Tr}\left(I_N \otimes S_{q_a,q_b,h_x} S^T_{q_a,q_b,h_x} \otimes I_2\right) = \mathrm{Tr}\left(S_{q_a,q_b,h_x} S^T_{q_a,q_b,h_x}\right),$$

we have

$$\|I_N \otimes S_{q_a,q_b,h_x} \otimes I_2\|_2 = \|S_{q_a,q_b,h_x}\|_2 \le c_1.$$

This proves the inequality. □

Lemma 3.3. *If* $\omega\sigma_0^2 \gg 1$ *then*

$$2\sigma_0\omega^2 \le \left\|\left(R^{(k)}\right)^{-1}\right\|_2 \le 2\sigma_1\omega^2, \quad k = 0, 1, 2, \ldots,$$

where σ_0, σ_1 *are defined in* (2.1).

Proof. Note that

$$\left(R^{(k)}\right)^{-1} = \mathrm{diag}\left(\left(P_1^{(k)}\right)^{-1}, \left(P_2^{(k)}\right)^{-1}, \ldots, \left(P_N^{(k)}\right)^{-1}\right),$$

where

$$\left(P_j^{(k)}\right)^{-1} = \mathrm{diag}\left(\left(P_{1,j}^{(k)}\right)^{-1}, \left(P_{2,j}^{(k)}\right)^{-1}, \ldots, \left(P_{N,j}^{(k)}\right)^{-1}\right), \quad j = 1, 2, \ldots, N.$$

Recall (2.3). We have

$$\left(P_{\ell,j}^{(k)}\right)^{-1} = \begin{pmatrix} 0 & 1/\beta_{\ell,j}^{(k)} \\ 1/\alpha_{\ell,j}^{(k)} & 0 \end{pmatrix}, \quad \ell,\ j = 1, 2, \ldots, N.$$

Therefore,

$$\left(P_{\ell,j}^{(k)}\right)^{-1}\left[\left(P_{\ell,j}^{(k)}\right)^{-1}\right]^T = \mathrm{diag}\left(1\Big/\left(\beta_{\ell,j}^{(k)}\right)^2, 1\Big/\left(\alpha_{\ell,j}^{(k)}\right)^2\right), \quad \ell, j = 1, 2, \ldots, N.$$

Now, recall that $0 < \sigma_0 \le u \le \sigma_1$. It follows immediately that

$$2\sigma_0\omega^2 = \sqrt{\max\left\{4\sigma_0^2\omega^4, 4/\sigma_1^2\right\}} \le \left\|\left(R^{(k)}\right)^{-1}\right\|_2 \le \sqrt{\max\left\{4\sigma_1^2\omega^4, 4/\sigma_0^2\right\}} = 2\sigma_1\omega^2$$

due to the proposition used. □

Theorem 3.4. *The eikonal splitting method* (2.12), (2.13) *is order two unconditionally asymptotically stable.*

Proof. We only need to verify (2.12) since the other case is similar. The underlying amplification matrix is

$$A^{(k)} = \left(I - \mu P^{(k)}\right)^{-1}\left(I + \eta Q^{(k)}\right). \tag{3.2}$$

Denote

$$T_1^{(k)} = I_N \otimes S_{q_a,q_b,h_x} \otimes I_2,\ T_2^{(k)} = S_{q_c,q_d,h_y} \otimes I_N \otimes I_2.$$

We deduce that

$$P^{(k)} = R^{(k)}T_1^{(k)},\ Q^{(k)} = R^{(k)}T_2^{(k)}.$$

A substitution of the above into (3.2) yields

$$\begin{aligned}A^{(k)} &= I + \left(I - \mu P^{(k)}\right)^{-1}\left(\mu P^{(k)} + \eta Q^{(k)}\right)\\ &= I + \left(I - \mu R^{(k)}T_1^{(k)}\right)^{-1}\left(\mu R^{(k)}T_1^{(k)} + \eta R^{(k)}T_2^{(k)}\right)\\ &= I + \left\{R^{(k)}\left[\left(R^{(k)}\right)^{-1} - \mu T_1^{(k)}\right]\right\}^{-1} R^{(k)}\left(\mu T_1^{(k)} + \eta T_2^{(k)}\right)\\ &= I + \left[\left(R^{(k)}\right)^{-1} - \mu T_1^{(k)}\right]^{-1}\left(\mu T_1^{(k)} + \eta T_2^{(k)}\right).\end{aligned}$$

Thus, according to Lemmas 3.2 and 3.3, we acquire subsequently that

$$\begin{aligned}
\left\|A^{(k)}\right\|_2 &= \left\| I + \left[\left(R^{(k)}\right)^{-1} - \mu T_1^{(k)}\right]^{-1} \left(\mu T_1^{(k)} + \eta T_2^{(k)}\right)\right\|_2 \\
&\le 1 + \left\| \left[\left(R^{(k)}\right)^{-1} - \mu T_1^{(k)}\right]^{-1} \left(\mu T_1^{(k)} + \eta T_2^{(k)}\right)\right\|_2 \\
&\le 1 + (c_1\mu + c_2\eta) \left\| \left[\left(R^{(k)}\right)^{-1} - \mu T_1^{(k)}\right]^{-1} \right\|_2 \\
&\le 1 + \frac{c_3\,(c_1\mu + c_2\eta)}{\omega^2} \left| \left\| \left(\frac{c_3}{\omega^2} R^{(k)}\right)^{-1} \right\|_2 - \left\| \frac{\mu c_3}{\omega^2} T_1^{(k)} \right\|_2 \right|^{-1} \\
&\le 1 + \frac{c_4\,(c_1\mu + c_2\eta)}{\omega^2}, \quad \omega \gg c_1\mu,
\end{aligned}$$

where c_1, c_2, c_3, c_4 are positive constants independent of ω [9]. This implies that

$$\left\|A^{(k)}\right\|_2 = 1 + O\left(1/\omega^2\right) \quad \text{as } \omega \to \infty.$$

The above ensures the theorem. □

4. Conclusions

The eikonal splitting method has been used intensively in practical laser beam computations, although its numerical stability is unclear. This has become a major obstacle in further developments and applications of the novel computational technology.

In this paper, we have investigated and proved the asymptotical stability of the latest eikonal splitting method (2.12) and (2.13) for solving paraxial Helmholtz equations modeling slowly varying envelope approximations of active light beams. This result remains to be true for problems with Dirichlet or Neumann boundary conditions.

The result acquired is not only new and straightforward, but also crucial to further algorithmic developments. It ensures the reliability and usability of the eikonal transformation associated with proper splitting, or decomposition, architectures. Similar approaches can be implemented for more sophisticated beam propagation simulations, such as those based on Maxwell's field equations when highly oscillatory waves are involved. The eikonal type BPM splitting extends the conventional concept of splitting methods. It has thrown light into the traditional numerical technique for the latest laser and light beam computations.

The matrix investigations utilized in this short note can also be extended for examining similar oscillation-free schemes, such as those for solving the Kukhtarev system in photorefractive material and laser–material interaction studies [8,3,4].

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